TOPIC 15: TWO PERSON GAMES (PAYOFF MATRIX)

We saw some how we could use tree diagrams in the last section to help with alternate move games and games against chance. Here we will introduce the use of a matrix array to help find strategies for games where both players decide on their strategy at the same time. In some games such as the hockey example in the previous example, it may be that the opponent's strategy does not become clear until a fraction of a second before you make your move. In this instance, it may not be enough to rely on reaction time to execute the best counterstrategy, it may be better to view this as a simultaneous move game.

Mathematical game theory was developed as a model of situations of conflict. Such situations and interactions will be called **games** and they have participants who are called players. In this section we will focus on simultaneous move games with **exactly two players**. Each player receives a payoff depending on the strategies chosen by both players. In **zero-sum games** one player's loss is the other player's gain and the payoff to both players for any given scenario adds to zero. For a basic introduction to game theory see Rolf [3] or Gilbert and Hatcher [2]. For a more detailed introduction see Straffin [4].

1. Pay-off matrix, Simultaneous Move games.

Definition 1.1. *The* **Payoff Matrix** *for a simultaneous move game is an array whose rows correspond to the strategies of one player (called the Row player) and whose columns correspond to the strategies of the other player (called the Column player). Each entry of the array (matrix) is the result, or* **payoff**, *obtained when the row player chooses the strategy corresponding to the row associated to the entry and the column player chooses the strategy corresponding to the column associated to the entry. This entry is often written as an ordered pair, where the first number represents the payoff for the Row player and the second number represents the payoff for the Column player. We assume that a larger payoff is better for each player.*

Example 1.1. *Two fitness chains Fitness Indiana and Get up 'n Go plan to expand by adding one fitness center to one of two neighborhoods neither of which have an existing fitness center. The first neighborhood has 5,000 people who would go to a local fitness center and the second neighborhood has 3,000 people who would use a local fitness center. If only one fitness center locates in a given neighborhood, that center gains all of the potential customers. If the stores locate in the same neighborhood, then 70% of the customers will go to Fitness Indiana, the better known chain. Each chain is fully aware of all of these details and must choose the neighborhood for its store without knowing the choice of its competitor.*

We can summarize the possible choices or strategies of each player and the corresponding payoffs in each possible scenario in a **Payoff matrix** as follows:

		Fitness	**Indiana**
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

This payoff matrix shows a pair of payoffs for each of the four possible scenarios. The first number in each pair is the number of customers that Get up 'n Go will have for that scenario and the second number in each pair shows the number of customers that Fitness Indiana will have in that situation. For example, if both business' locate in the First Neighborhood, Fitness Indiana will get 70% of the customers in that neighborhood (70% of 5000 = 3500) and Get up 'n Go will have the remaining $5000 - 3500 = 1500$ customers.

1.1. **Constant-sum and Zero-sum games.** We will concentrate on zero-sum games in this course.

Definition 1.2. *A two person* **zero-sum game** *is a game where the pair of payoffs for each entry of the payoff matrix sum to 0.*
This means that one player's gain is equal to the other player's loss on any given play of the game.

Definition 1.3. *A two person* **constant-sum game** *is a game where the pair of payoffs for each entry of the payoff matrix sum to the same constant C.*
The analysis of these games is the same as that of zero sum games, since subtracting the given constant from the column player's payoffs makes it a zero sum game.

By Convention the payoff matrix for a two player zero-sum game or a two player constant-sum game, shows the strategies for both players with the payoffs for the **row player** only as entries. The payoffs for the column player for each situation can be deduced from the row player's payoff.

Example 1.2. Example: Rock Paper Scissors *In the game of Rock-scissors-paper, the players face each other and simultaneously display their hands in one of the following three shapes: a fist denoting a rock, the forefinger and middle finger extended and spread so as to suggest scissors, or a downward facing palm denoting a sheet of paper. The rock wins over the scissors since it can shatter them, the scissors wins over the paper since they can cut it, and the paper wins over the rock since it can be wrapped around it. The winner collects a penny from the opponent and no money changes hands in the case of a tie.*

The payoff matrix for this game is shown below:

		Colleen		
		Rock	**Paper**	**Scissors**
Roger	**Rock**	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Example 1.3. *(Two Finger Morra) Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player's call matches the total, no money changes hands.*

(a) Write down a pay-off matrix for this game (here the strategy $(1, 2)$ means that the player holds up one finger and shouts 2).

Charlie

		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$
R	$(1,2)$				
u	$(1,3)$				
t	$(2,3)$				
h	$(2,4)$				

1.2. **Using Expected value or averages as payoff.** Sometimes the pay-off for each combination of strategies might be a random variable, in this case we could use the expected payoff or average payoff to calculate the payoff for the row player in a zero sum game.

Example 1.4. Football Run or Pass? [Winston [5]] (Using averages as payoffs) *In football, the offense selects a play and the defense lines up in a defensive formation. We will consider a very simple model of play selection in which the offense and defense simultaneously select their play. The offense may choose to run or to pass and the defense may choose a run or a pass defense. One can use the average yardage gained or lost in this particular League as payoffs and construct a payoff matrix for this two player zero-sum game. Lets assume that if the offense runs and the defense makes the right call, yards gained average out at a loss of 5 yards for the offense. On the other hand if offense runs and defense makes the wrong call, the average gain is 5 yards. On a pass, the right defensive call usually results in an incomplete pass averaging out to a zero yard gain for offense and the wrong defensive call leads to a 10 yard gain for offense. Set up the payoff matrix for this zero-sum game.*

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**		
	Pass		

Example 1.5. *In a sabre match in fencing, each fencer can choose to attack straight off the line (A) when the referee gives the signal to begin or the fencer can hold back (H), delaying their attack or taking a defensive position. Bouts are three minutes long, and are fenced to five points. If no fencer reaches five points, then the one with the most points after three minutes wins. Lets assume Rhonda and Cathy are opponents in a saber match. Each interaction can result in a point for Rhonda (payoff of 1 for Rhonda), no points for either player(Payoff of 0 for Rhonda) or a point for Cathy (Payoff of -1 for Rhonda). No single payoff is guaranteed for any set of strategies, but Rhonda can calculate the probability that each situation will result in one of the payoffs (for Rhonda) by collecting statistics and observing videos of her opponents.*

Suppose Rhonda has estimated the probability distributions shown below for her payoffs in each situation, use expected payoffs for Rhonda to construct Rhonda's Payoff matrix for her saber match against Cathy. (Note that since Rhonda's gain is always Cathy's loss, Rhonda's expected payoff is always the negative of Cathy's expected payoff).

R(A), C(H)

Payoff	Probability
1	0.3
0	0.2
-1	0.5

R(A), C(A)

Payoff	Probability
1	0.7
0	0.1
-1	0.2

R(H), C(A)

Payoff	Probability
1	0.3
0	0.1
-1	0.6

R(H), C(H)

Payoff	Probability
1	0.7
0	0.1
-1	0.2

		Cathy	
		A	**H**
Rhonda	**A**		
	H		

1.3. **Using percentages or probabilities as payoffs; a constant sum game.** In a win-loss situation, we can use the probability of a win as the payoff for the row player. This gives us a constant sum game where the probably of a win for both players adds to 1. We can also use percentages as payoffs in a similar way.

Example 1.6. Dutta [1] **Drug Testing** *Suppose two swimmers, Rogers and Carter, are about to compete in a runoff. Each athlete has the option of using a performance enhancing drug (d) or not using it (n). Lets assume that both competitors have equal abilities and are the two top competitors with no serious competition for first and second place.*

(a) If no drug testing exists we give each a fifity percent chance of winning the race if neither takes the drugs. Suppose on the other hand that one takes the drug and one does not, then the one who takes the drug is sure to win. Finally if both take the drug, each has a fifty percent chance of winning. Use the probability of a win for Rogers to complete the payoff matrix shown below where d denotes the strategy of taking the drug and n denotes the strategy of not taking the drug.

No Testing

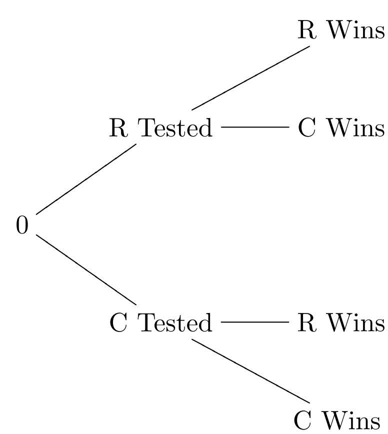
		Carter	
		d	**n**
Rogers	**d**		
	n		

(b) **How Drug Testing Changes The Game** On the other hand lets consider the situation where the IOC tests (only) one of the swimmers and that both swimmers are equally likely to be tested. If a swimmer tests positive, then the race is awarded to the other swimmer and the swimmer who tested positive faces a further penalty. This penalty would have a large impact on the swimmer's career, we will denote its (payoff) value by $-b$ where b is a relatively large positive number. Using 1 as the payoff for winning the race and (-1 for not winning) fill out the probability distributions for each player's payoff for each scenario shown below and calculate his expected payoff for each situation.

The tree diagrams should help to calculate probabilities in each case :

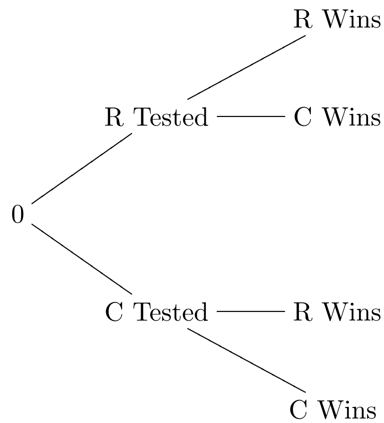
R(d), C(d)

	Payoff R	Payoff C	Probability
RT, RW	-b	1	
RT, CW	-b	1	
CT, RW	1	-b	
CT, CW	1	-b	



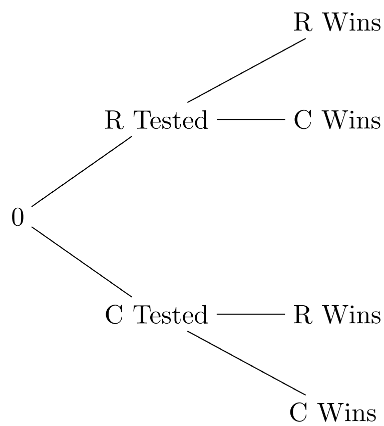
R(d), C(n)

	Payoff R	Payoff C	Probability
RT, RW	-b	1	
RT, CW	-b	1	
CT, RW	1	0	
CT, CW	0	1	



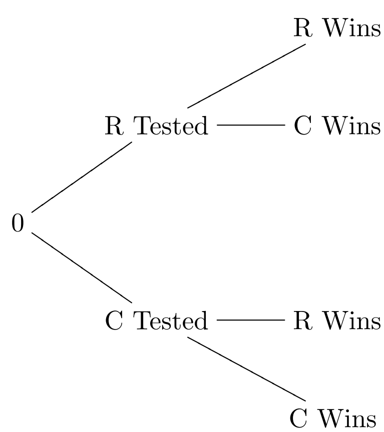
R(n), C(d)

	Payoff R	Payoff C	Probability
RT, RW	1	0	
RT, CW	0	1	
CT, RW	1	-b	
CT, CW	1	-b	



R(n), C(n)

	Payoff R	Payoff C	Probability
RT, RW	1	0	
RT, CW	0	1	
CT, RW	1	0	
CT, CW	0	1	



(a) Calculate the expected payoffs for each player in each of the four possible scenarios.

IOC Testing

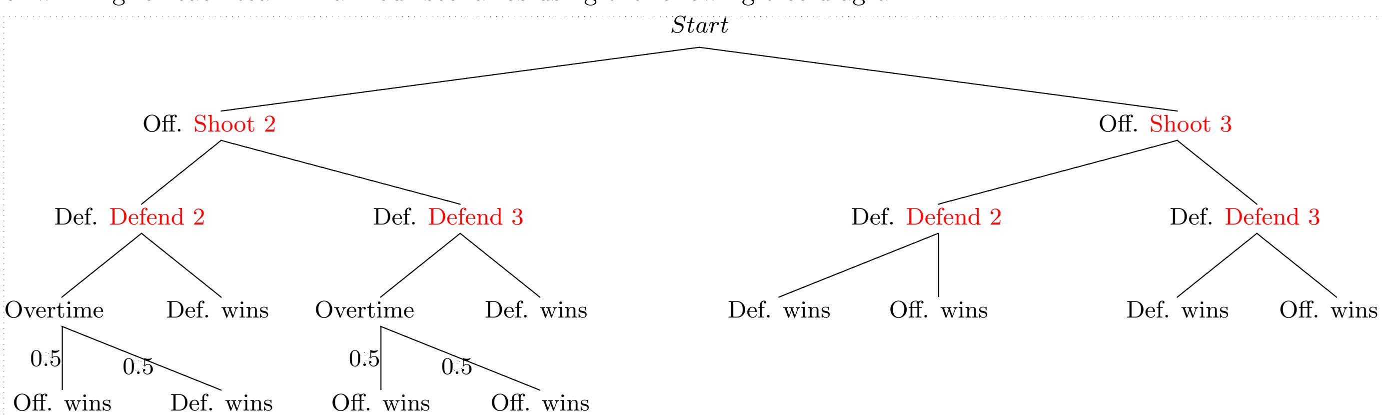
		Carter	
		d	**n**
Rogers	**d**		
	n		

Endgame Basketball [Ruminski] Often in late game situations, a team may find themselves up by two points with the shot clock turned off. In this situation, the offensive team must decide whether to shoot for two points, hoping to tie the game and win in overtime, or to try for a three pointer and win the game without overtime. The defending team must decide whether to defend the inside or outside shot. we assume that the probability of winning in overtime is 50% for both teams.

In this situation, the offensive team's coach will ask for a timeout in order to set up the play. Simultaneously, the defensive coach will decide how to set up the defense to ensure a win. Therefore we can consider this as a simultaneous move game with both coaches making their decisions without knowledge of the other's strategy. to calculate the probability of success for the offense, Ruminski uses League wide statistics on effective shooting percentages to determine probabilities of success for open and contested shots. He gets

Shot	Success rate
open 2pt.	62.5%
open 3pt.	50%
Contested 2pt.	35.7%
Contested 3pt.	22.8%

Using this and the 50% probability of winning in overtime for each team, we can figure out the probability of winning for each team in all four scenarios using the following tree diagram:



(a) Use the above percentages to fill in the probabilities where appropriate on the tree diagram above.

(b) Use those probabilities to fill in the probabilities of a win for the row player (offense) in the payoff matrix below. (Note the probability for a win for the defense team is 1 - prob. win for offense.)

		Defending	Team
		Defend 2	**Defend 3**
Offense	**Shoot 2**		
	Shoot 3		

References

1. Prajit K. Dutta, *Strategies and games*, The M.I.T. Press.
2. Hatcher R Gilbert, G, *Mathematics beyond the numbers*, Wiley (2000).
3. Howard Rolf, *Finite mathematics*, Cengage Learning.
4. Phillip D. Straffin, *Game theory and strategy*, The Mathematical Association of America.
5. Wayne Winston, *Mathletics*, Princeton University Press.